# Erdős-Ko-Rado theorem, Erdős Matching Conjecture, between and around

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Let $[n] = \{1, 2, \ldots, n\}$ be our underlying set. $\binom{[n]}{k}$ will denote the family of all $k$-element subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called intersecting if any pair of its members have a non-empty intersection. The celebrated theorem of Erdős, Ko and Rado determines the maximum size of an intersecting family of $k$-element subsets: If $2k \leq n, \mathcal{F} \subset \binom{[n]}{k}$ is intersecting then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Another problem raised by Erdős was to determine the largest family $\mathcal{F} \subset \binom{[n]}{k}$ such that it contains no $\ell$ pairwise disjoint members. He proved that for sufficiently large $n$ the best is to choose all sets meeting a fixed $\ell-1$-element set, so the maximum of $|\mathcal{F}|$ is $\binom{n}{k} - \binom{n-\ell+1}{k}$. He conjectured that either this family is the largest or $\binom{[\ell k-1]}{k}$. This was proved for smaller and smaller $n$ during the years, but it is still not completely solved.

Choose an integer $\ell \geq 2$ and take the following sum.

$$\sum_{1 \leq i < j \leq \ell} |F_i \cap F_j|. \tag{1}$$

If $\mathcal{F}$ is intersecting then every term here is at least 1, therefore the total sum is at least $\binom{\ell}{2}$. Does this weaker condition $\binom{\ell}{2} \leq \sum_{1 \leq i < j \leq \ell} |F_i \cap F_j|$ imply the upper bound of EKR? Much more is true for large $n$. We (Frankl, Katona, Kartal Nagy) recently proved that $\binom{\ell}{2}$ can be replaced by $\binom{\ell-1}{2} + 1$. However $\binom{\ell-1}{2}$ is not good enough, as a construction shows containing more than $\binom{n-1}{k-1}$ sets. We proved (jointly with Jian Wang) that this construction is the best possible for large $n$. The condition of the Erdős Matching Conjecture can also be formulated in our new language: (1) is not zero. Some newer similar results are introduced connecting the two problems.